Happy Endings and New Beginnings: An Unsolved Problem in Discrete Geometry

by Jessica Shand

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Abstract: This expository paper concerns the history and developments of the Erdos-Skezeres conjecture and what has been dubbed the happy ending problem, which has remained unsolved since 1935. Against the backdrop of classical Euclidean geometry, we use the problem as a gateway into the study of geometrical objects and properties that are combinatorial in nature and representation. To conclude, we examine the most recent results toward improving upper and lower bounds for $N_0(n)$, the minimum number of points in general position needed to guarantee the admission of a convex n-gon.

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1 The Original Problem

In the winter of 1932-1933, the Hungarian mathematician Esther Szekeres (then Esther Klein) made the following observation [**2**]:

Proposition (Klein). *From 5 points of the plane of which no three lie on the same straight line, it is always possible to select 4 points determining a convex quadrilateral.*

That is, suppose we choose a set of 5 points in $\mathbb{R}^2$, of which no three are collinear. (We say that the points are in *general position*). Then according to Klein's proposition, some subset of 4 of these points is guaranteed to determine a *convex quadrilateral*, which we define as *a set of four points in convex position* as per the following terminology [**1**]:

Definition 1. *In the Euclidean plane, a set C is convex if for every pair of points $(x_1, y_1), (x_2, y_2) \in C$, every point on the straight line segment that joins (x_1, y_1) and (x_2, y_2) is also in C.*

Definition 2. *The convex hull of a set X of points is the intersection of all convex sets in $\mathbb{R}^2$ that contain X; hence it is the smallest convex set that contains X, and we call it conv(X).*

Definition 3. *A set X of points is in convex position if for every $x \in X$, x is a vertex of conv(X).*

Equipped with this terminology, we can now prove Klein's proposition about convex quadrilaterals before proceeding to the more general problem for n-gons given arbitrary n.

Proof. First, we show that 4 points are not sufficient to guarantee a convex quadrilateral. To this end, we need only construct one example of a 4-point set in general position that does not contain a convex quadrilateral. Consider the set of points $\{(0, 2), (2, 2), (1, 1), (1, 0)\}$. As can be seen in Figure 1.1, no three of these points are collinear. However, no convex quadrilateral can be formed using these points as vertices.

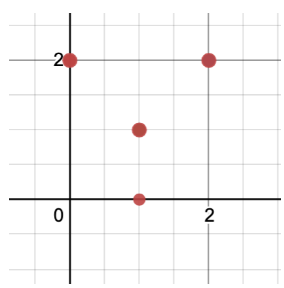

Figure 1.1: A set of 4 points in general position that do not contain a convex quadrilateral.

Now we proceed to examine a 5-point set X in general position. Configurations of conv(X) cannot have less than three vertices since conv(X) is itself a convex polygon. Thus there are three possible configurations of conv(X) that respectively take 5, 4, and 3 points of X as vertices, as illustrated in Figure 1.2 below. The remaining points of X lie in the interior of conv(X) [**1**].

Starting with the first case, if conv(X) is a pentagon, then X is in convex position and we can choose any 4 of the points to form a convex quadrilateral. On the other hand, if 4 points of X are vertices of conv(X) with the remaining point in the interior, then the four vertices form a convex quadrilateral. Lastly, if conv(X) is a triangle taking three points of X as its vertices with the remaining two points in its interior, then the line l through the two interior points must intersect two of the sides of the triangle. We can then take the side not intersected by l (which we call S) together with l and the lines containing the points on S and the interior points of the triangle as our convex quadrilateral.

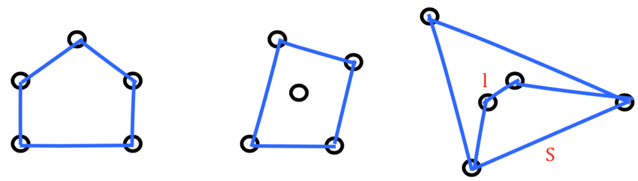

Figure 1.2: From left to right, a configuration of conv(X) taking all 5 points of X as its vertices; a configuration taking 4 points as vertices; a configuration taking 3 points as vertices.

Hence we have shown that any set of 5 points in general position in the Euclidean plane is guaranteed to admit a convex quadrilateral. Since any fewer points will not guarantee a convex quadrilateral, we have drawn the desired conclusion. □

2 Generalization for Arbitrary n

After presenting the above observation at a meeting of the Anonymous Society, an informal group of eager mathematics students in downtown Budapest in the 1930s [**6**], Klein proceeded to pose the following question [**2**].

Let $N_0(n)$ represent the minimum number of points in general position in the plane required to guarantee that some subset of n of those points form a convex n-gon. Given an arbitrary natural number n, does $N_0(n)$ exist? If so, how is $N_0(n)$ determined as a function of n?

Two of Klein's colleagues at the time, Paul Erdos and George Szekeres, took an intensive interest in this more general problem, leading to the marriage of Klein and Szekeres in 1937 – and in turn inspiring Erdos to refer to it as the "happy ending" problem [**6**]. Erdos and Szekeres managed to prove that the first question was to be answered in the affirmative [**2**], leading to the formulation of the following theorem:

Theorem (Erdos-Szekeres). *For any positive integer $n \geq 3$, there is an integer $N_0(n)$ such that any set of at least $N_0(n)$ points in the plane in general position contains n points that form a convex polygon.*

Erdos and Szekeres provided two proofs for this theorem, one of which is presented in the following section. Moreover, they formulated but never managed to prove the following conjecture [**3**]:

Conjecture (Erdos-Szekeres). $N_0(n) = 2^{n-2} + 1$.

According to Erdos, it was E. Makai and P. Turan who had first managed to show that one would need nine points to guarantee the admittance of a convex pentagon [**2**], but a proof did not appear in the mathematics literature until J.D. and J.G. Kalbfleisch and R.G. Stanton of Louisiana State University published one in 1970 [**4**]. In 2006, L. Peters, a student of Szekeres, proved that $N_0(6) = 17$ [**8**]. But as we approach the centenary of the publication of Erdos and Szekeres' seminal article "A combinatorial problem in geometry," first published in *Compositio Mathematica* in 1935 [**2**], a proof of the conjecture for general n has yet to be produced [**1**].

3 Proof of the Existence of $N_0(n)$

Trivially, we know that $N_0(3) = 3$ because any triangle in Euclidean space is convex, and any less than 3 points is not sufficient to form any polygon [**1**]. We have proven that $N_0(4) = 5$. We have stated results for incrementally increasing values of n. Our motivating question, then, is this [**1**]: is it possible that there exists some larger n for which, regardless of the number of points we have, we still cannot admit a convex n-gon? Why should we believe that as n continues to increase, we will always find a value for $N_0(n)$?

The first proof for the main theorem of Erdos-Szekeres, which is combinatorial in flavor, has been hailed as one of the early demonstrations of the elegance and power of the Ramsey Principle [**6**]. However, as Erdos and Szekeres admitted in the original paper, its resulting upper bound on $N_0(n)$ is very poor, giving $N_0(5) = 21$ [**2**], more than twice the correct value [**4**]. Thus we present their second proof for the existence of $N_0(n)$, which gives a better upper bound. To start, we include a few basic definitions [**2**].

Definition 4. *A sequence of n line segments is consecutive if the right endpoint of the ith line segment is the left endpoint of the $(i+1)$th line segment for all $1 \leq i < n$.*

Definition 5. *A sequence of n consecutive line segments is a cap if, for all $1 \leq i < n$ line segments, the slope of the ith line segment is greater than that of the $(i+1)$th line segment. In other words, the slopes of the line segments are monotonically decreasing.*

Definition 6. *A sequence of n consecutive line segments is a cup if the slopes of the line segments are monotonically increasing.*

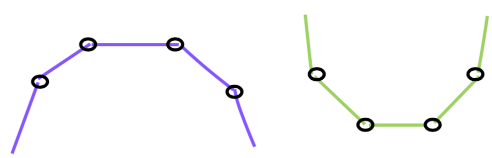

Figure 3.1: An example of a cap (left) and an example of a cup (right).

Equipped with these notions, we proceed to begin the proof by letting $f(k,l)$ represent the smallest number of points such that any collection of $f(k,l)$ points on the plane contains a cap of length k or a cup of length l.

Proof. First, note that $N_0(n) \leq f(n,n)$ because if we have a cap or a cup of length n, then by connection of the first and last points, the n points determine a convex n-gon.

Now, observe that $f(k,3) = k$. Given any k points $x_1, \ldots, x_k$, if the k points form a cap, then it is a cap of length k and we are done. If the k points do not form a cap, then there must exist some consecutive pair of line segments $i, i+1$ for which the slope of line segment i is greater than that of $i+1$, which implies that these two segments form a cup of length 3. A similar argument shows that $f(3,k) = k$, which gives us $f(3,k) = f(k,3) = k$.

We proceed by showing the following recurrence: $f(k,l) \leq f(k-1,l) + f(k,l-1) + 1$.

Considering a collection X of $f(k-1,l) + f(k,l-1) + 1$ points in the plane, if some k points form a cap, then the cap is of length k and we are done. On the other hand, if we assume that there are no caps of length k, define a subset Y of X containing any existing $(k-1)$-caps of X. Let L represent the set of leftmost endpoints of caps in Y. Considering $\{X \setminus Y\}$, if the length of this set is greater than or equal to $f(k-1,l)$, we are done because it must admit a cup of length l and caps must have length less than $k-1$. On the other hand, if $|X \setminus Y| < f(k-1,l)$, then $|L| > f(k,l-1) - 1 \implies |L| \geq f(k,l-1)$. Since X does not have any caps of length k by assumption, we know that L must admit a cup of length at least $(l-1)$. Since the points of this cup are the leftmost endpoints of a $(k-1)$-cap, then either the cap or the cup can be elongated by one depending on the slope of the line between the common point, the second-rightmost point of the $(k-1)$-cap, and the second-leftmost point of the $(l-1)$-cup. Hence we have come to the desired conclusion.

By applying the identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, we can see that
$f(k,l) \leq f(k-1,l) + f(k,l-1) - 1$ (from before)
$\leq \binom{k+l-3}{k-3} + \binom{k+l-3}{k-2} - 1$
$= \binom{k+l-2}{k-2} + 1.$

Setting $k = l = n$, we have $f(k,k) = \binom{2k-4}{k-2} + 1$. By connection of the first and last points, every set of k convex or concave points determines a convex k-gon, so $[\binom{2k-4}{k-2} + 1]$ points always contain a convex k-gon. Moreover, as in every convex $(2k-1)$-gon there is always either a convex or concave configuration of k points, it is possible to give $\binom{2k-4}{k-2}$ points such that no convex $(2k-1)$-gon can be selected. $\square$

Thus we have not only confirmed that $N_0(n)$ exists for all values of n, but in particular we have found the upper bound on $N_0(n)$ first published in Erdos and Szekeres' findings [**2**].

4 Progress Towards the Upper and Lower Bounds

Given that $N_0(n)$ exists for all $n \geq 3$, the next goal is to improve the upper and lower bounds for $N_0(n)$ so as to either prove or disprove the Erdos-Szekeres Conjecture, a problem that has not been solved for almost a century [**1**]. However, noteworthy results by mathematicians across the globe, which we document here, have slowly but surely pointed towards better upper and lower bounds for $N_0(n)$.

The results of the geometric proof from the previous section initially set the best upper bound for $N_0(n)$ at $[\binom{2n-4}{n-2} + 1]$ [**2**]. This was improved slightly by Chung and Graham [**10**], who showed in 1998 that $N_0(n) \leq \binom{2n-4}{n-2}$, and then again by Kleitman and Pachter [**5**], who showed in the same year that $N_0(n) \leq \binom{2n-4}{n-2} + 7 - 2n$. Toth and Valtr improved this slightly to $N_0(n) \leq \binom{2n-5}{n-3} + 2$ [**9**], continuing in the series of marginal improvements until a major breakthrough was made by A. Suk in 2016 [**7**], which stands as the best upper bound:

$$N_0(n) \leq 2^{n+6n^{2/3}\log n}.$$

As for the lower bound on $N_0(n)$, Erdos and Szekeres used the results of the proof from the previous section to construct sets of 2^{n-2} points to prove that $N_0(n) \geq 2^{n-2} + 1$ in a paper published in 1961 [**3**]. Given this result, if it is found that $N_0(n) \leq 2^{n-2} + 1$, then we will have an equality and the general conjecture will have been proven true. As of the writing of this paper, such a feat has not been achieved.

5 References

[1] J. Bedich. *An Introduction to the Happy Ending Problem and the Erdos-Szekeres Conjecture.* Ohio State University, 2018.

[2] P. Erdos and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463-470, 1935.

[3] P. Erdos and G. Szekeres. On some extremum problems in elementary geometry. *Ann. Univ. Sci. Budapest. Eotvos Sect. Math.*, 3-4:53-62, 1960/1961.

[4] J. D. Kalbfleisch, J. G. Kalbfleisch, and R. G. Stanton. A combinatorial problem on convex n-gons. *Proc. Louisiana Conf. on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1970)*, 180 - 188, 1970.

[5] D. Kleitman and L. Pachter. Finding convex sets among points in the plane. *Discrete Comput. Geom.*, 19(3):405 - 410, 1998.

[6] A. Soifer. *The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators.* New York: Springer, 2009.

[7] A. Suk. On the Erdos-Szekeres convex polygon problem. *J. Amer. Math. Soc.*, 2016.

[8] G. Szekeres and L. Peters. Computer solution to the 17-point Erdos-Szekeres problem. *ANZIAM J.*, 48(2):151-164, 2006.

[9] G. Toth and P. Valtr. Note on the Erdos-Szekeres theorem. *Discrete Comput. Geom.*, 19(3):457-459, 1998.

[10] F. R. K. Chung and R. L. Graham. Forced convex n-gons in the plane. *Discrete Comput. Geom.*, 19(3):367-371, 1998.