GAME MUSIC:
Game-Theoretic Principles in Music Composition

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Abstract

The history of the application of mathematical concepts in musical composition and theory dates back to at least Greek antiquity. However, it was only in the twentieth century that a handful of composers of contemporary classical music, largely led by Iannis Xenakis, began to formalize sophisticated stochastic methods within their work. In this paper, we survey two pieces of game music within the contemporary classical repertoire: Xenakis's own *Achorripsis* – a group improvisation based on a matrix generated by a Poisson distribution – and a more recent Xenakis-inspired installation by Davide Morelli and Marco Liuni, a zero-sum game between opposing players. We then proceed to design and implement our own set of musical games related directly to the tension between determinism and stochasticity in Xenakis, Morelli, and Liuni's work, concluding that the delicate balance between the two is the foundation of a score's success.

Introduction

Greek-French composer Iannis Xenakis (1922 - 2001) is widely recognized and praised as a pioneer of the twentieth century musical avant-garde [3]. Originally trained as an architect under Charles-Edouard Jeanneret (better known as Le Corbusier), Xenakis took up composition under Olivier Messiaen, who helped him foster a musical aesthetic driven by mathematical modeling [5]. His output included concert pieces for orchestra, a handful of electroacoustic works, and numerous theoretical writings that dealt with set theory, stochastic processes, and physics – among other academic subjects – as applied to music, revolutionizing contemporary concepts of sound composition [4].

Regarding his singular musical aesthetic, the composer has said that "what is obtained by calculation always has limits. It lacks inner life, unless very complicated techniques are used. Mathematics gives structures that are too regular and that are inferior to the demands of the ear and the intelligence. The great idea is to be able to introduce randomness in order to break up the periodicity of mathematical functions, but we're only at the beginning" [8]. For the purpose of this project, then, in the introductory section we focus on the element of chance as manifested in Xenakis's *Achorripsis* (1957) for twenty-one instruments [10]. We also take a look at a Xenakis-inspired installation conceived by Marco Liuni and Davide Morelli in 2006 before designing and implementing game-theoretic principles in our own original musical composition [6].

Achorripsis *for 21 Instruments (1957)*

As aforementioned, the implementation of creative game strategies appears throughout Xenakis's oeuvre. Arguably the best-known uses of game theory as a basis for musical composition in all of contemporary classical music occur in Xenakis's *Duel* (1959) and *Stratégie (1962)*, each scored for two orchestras constituting the opposing players of a game [2].

A predecessor to *Duel* and *Stratégie, Achorripsis* (which translates to "jets of sound") utilizes a matrix *M* to define the sequence of events that constitutes the piece. In particular, and as is detailed in Xenakis's work *Formalized Music: Thought and Mathematics in Composition*, the piece invokes the Poisson formula to determine a distribution of zero events, single events, double events, triple events, and quadruple events across 196 cells [9]. In this formulation, a "zero event" corresponds to silence while a "quadruple event" corresponds to the greatest possible *density* of sounds, where density is the number of sounding tones per time interval [1].

Conceived of by mathematician Siméon Denis Poisson (1781-1840), the Poisson distribution models the number of successes out of a series of independent Bernoulli trials in a given time interval for which the average time between events is known and the probability of success p is small. Xenakis was familiar with the distribution's applications to various time- and space- oriented situations in which events are rare, such as the occurrence or quantity of precipitation in an area, the number of sales in a day, or the number of persons affected by an uncommon disease in a large population [7].

Returning to Xenakis, then, note that the choice of the number 196 as the number of cells in *M* was, according to the composer, arbitrary; Xenakis also made the choice to set $\lambda = 0.6$ in the Poisson formula for "convenience in calculation" [9]. In Poisson's law, X denotes the random variable for which the values k are possible, P is the probability of occurrences, and λ is the mean number of occurrences in an interval:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \tag{1}$$

In determining how to distribute events of varying sonic density across the cells, Xenakis initially substituted the integer values $k = 0$ through $k = 5$ in (1), using $\lambda = 0.6$. However, after seeing that the probability of the quintuple event was statistically insignificant (indeed, it would not likely appear in a sample size of 196 events), he resolved to set the quadruple event as the maximum possible density [1]. After rounding, Xenakis thus arrived upon the following probabilities and expected numbers of occurrences for each event, which have been confirmed by the author:

Occurrences of Sonic Densities in *Achorripsis*

***Value of* k**	**P (X = k)**	***# of occurrences***
0	0.5488	107
1	0.3293	65
2	0.0988	19
3	0.0198	4
4	0.0030	1
TOTAL		**196**

Table 1. The probability and number of occurrences of each possible sonic density in Xenakis's 196-cell sample, utilizing a Poisson distribution with $\lambda = 0.6$.

From here, Xenakis again made the arbitrary choice to divide the matrix *M* into 7 rows (denoting the instrumentation of piccolo/clarinet/bass clarinet, oboe/bassoon/contrabassoon, string glissando, string pizzicato, bowed strings, xylophone/wood block/bass drum, and trumpets/trombone) and 28 columns (denoting units of time, each approximately 6.5 measures in length) [1]. He then reapplied Poisson's law for each event type, first iterating across columns and then across rows, each of which requires a recalculation of λ; simply, λ is equated to the quotient of each number in the right-most column of Table 1 and the number of columns or rows, respectively [9].

Without expanding upon the remaining calculations, one should note that the final calculation of *M*, though fundamentally rooted in the Poisson distribution, is not entirely dictated by it. Besides the arbitrary division into 196 cells, consisting of 7 rows and 28 columns, Xenakis intervenes to suit his scheme as needed, a crucial component of the resulting score [1]:

Game Matrix *M* for *Achorripsis*

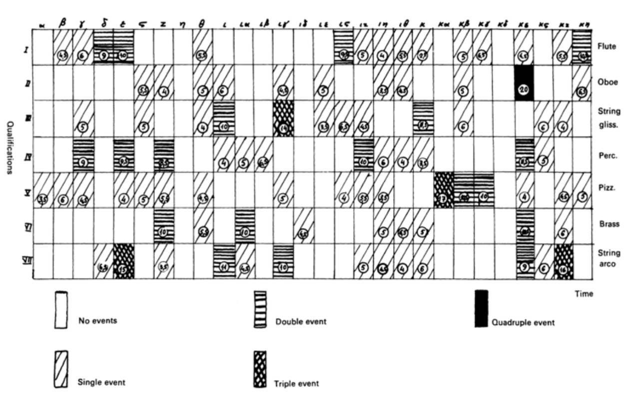

Figure 1. The final scheme for *Achorripsis*'s matrix *M* as it appears in [9].

Liuni and Morelli's Playing Music *(2006)*

In the 2006 Xenakis-inspired score for *Playing Music*, Marco Liuni of the Conservatorio di Musica B. Marcello and Davide Morelli of the Università di Pisa take interest in Xenakis's notion of external conflict between musical structures [6]. In Liuni and Morelli's score, players follow a game matrix (similar to *Achorripsis)*, but the introduction of this concept of conflict grants agency to each individual and constructs a fair zero-sum game in which one player wins [6]. Moreover, the researchers implement a computer software that renders Xenakis's orchestral ensembles as electronic orchestras controlled by two conductors, one for each orchestra [6].

In this construction of the musical game, each two-move combination is associated with a musical event consisting of pre-recorded sound samples. After flipping a coin to determine which player will move first, each of the two players chooses between 6 possible moves to generate a couplet of numbers *(m,n)* corresponding to an entry in a 6 x 6 game matrix. The software then gives the value to the first player if the entry is positive, or to the second player if the entry is negative [6]. Players make moves within a time limit that decreases as the game goes on, by moving in front of a camera in one of six ways. The game ends in one of three ways: (1) a certain number of moves are completed, (2) a certain duration of time has passed, or (3) a player's score reaches some predetermined amount [6]. As a tribute to Xenakis's own compositions, Liuni and Morelli demand that the winner be awarded some sort of prize in order to create the incentive to win and to "enforce game sensation" [9].

Game Matrix for *Playing Music*

	I	II	III	IV	V	VI
I	-13	15	43	-13	15	-13
II	15	-13	-13	-13	15	-13
III	43	-13	-41	71	15	-41
IV	-13	43	43	-13	-13	-13
V	15	-13	15	15	-13	-13
VI	-13	-13	-41	-13	-13	43

Table 2. A possible game matrix for Liuni and Morelli's *Playing Music*; for ease of performance, numerical values can be replaced by g++, g+, g, p+, and p (in descending order from highest positive to lowest negative value).

Having first briefly surveyed Xenakis's *Achorripsis* and then Marco Liuni and Davide Morelli's *Playing Music*, we have seen that each is simultaneously deterministic and stochastic in its own ways. While the former allows the composer some degree of freedom in designing the rules of the game and its score leaves the execution of "clouds of sound" up to interpretation and improvisation, the matrix M is completely determined prior to performance. On the other hand, the possible sonic events are predetermined in the latter, as they are inputted as sound samples within the software; the flip of the coin to determine move order introduces an element of chance, as well as the ability of players to interact and use the game matrix at their own will. With this tension in mind, we proceed to execute a series of musical games of our own design.

Methods

Following the method attributed to *Achorripsis,* we create a slightly smaller 5 x 10 matrix N, retaining the use of the Poisson formula, now with $\lambda = 0.5$, to determine sound densities across cells. However, following *Playing Music*, we now introduce randomness in two crucial ways:

(1) Rather than fixing time intervals represented by columns at approximately 6.5 measures of length, we generate a random integer between 1 and 60 to determine the length of each interval in seconds, to be applied across each timbre of sound for the given column.

(2) To determine the timbres of sound utilized in the score, we roll a die five times each time the game is restarted. Timbres may be repeated or may not occur at all, and are assigned sequentially to each of the five rows as follows:

Instrumentation for the Matrix N_i

Outcome	Timbre
I	Aeolian/wind sounds (flute)
II	Percussive noises
III	Plucked strings (guitar)
IV	Speech/vocalized noises
V	Humming
VI	Pedalled chords (piano)

Table 3. The scheme for introducing randomness into the instrumentation of our piece. Entries in the first column denote the outcome of a die roll; entries in the second column denote the corresponding timbre of sound to be used.

Additionally, in a live setting in which players improvise simultaneously, each corresponding to one timbre of sound, players may choose to individually proceed to the next cell at their own discretion, but may not return to a previous cell and must still adhere to the randomly generated durations of each column.

After the score is read from beginning to end, a winner is determined on the basis of sound density across the duration of the piece, defined negatively as the absence of silence of the corresponding timbre. (Note that while we retain the zero, double, triple, and quadruple event schematic, which is explicitly written into the score, the interpretation and execution of each event is left to the performers).

We iterate this process three times, generating a new score with matrix N_1, N_2, N_3 for each iteration respectively. Note that the reduction in number of cells from 196 in Xenakis's matrix M to 50 for our matrices N_i results in statistical insignificance for the quadruple event, which is thus discarded in the generation of the score:

Occurrences of Sonic Densities for the Matrix *N*

***Value of* k**	**P (X = k)**	***# of occurrences***
0	0.6065	30
1	0.3033	15
2	0.0758	4
3	0.0126	1
4	0.0016	0
TOTAL		**50**

Table 4. The probability and number of occurrences of each possible sonic density a 50-cell sample, utilizing a Poisson distribution with $\lambda = 0.5$ and rounded to nearest ten-thousandth (for probabilities) or integer (for occurrences).

Per Xenakis, we reapply Poisson's formula to determine the distribution of densities across 50 cells, first for the 10 columns and then for the 5 rows of the N_i:

Poisson Distribution of Zero Events

Value of *k*	***P (X=k)* for col**	**P_k x 10 col**	***P (X=k)* for rows**	**P_k x 5 rows**
1	.1494	1	.0149	0
2	.2240	2	.0446	0
3	.2240	2	.0892	0
4	.1680	2	.1339	1
5	.1008	1	.1606	1
6	.0504	1	.1606	--
7	.0216	0	.1377	--
8	.0081	0	.1033	--
9	.0027	0	.0688	--
10	.0008	0	.0413	--

Table 5. For zero events, we calculate that $\lambda = \frac{30}{10} = 3$ for columns and $\lambda = \frac{30}{5} = 6$ for rows. Values were rounded to nearest ten-thousandth (for probabilities) or integer (for occurrences).

Poisson Distribution of Single Events

Value of *k*	***P (X=k)* for col**	**P_k x 10 col**	***P (X=k)* for rows**	**P_k x 5 rows**
1	.3347	3	.0149	0
2	.2510	3	.2240	1
3	.1255	1	.2240	1
4	.0471	0	.1680	1
5	.0141	0	.1008	1
6	.0035	0	.0504	--
7	.0008	0	.0216	--

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8	.0001	0	.0081	--
9	--	--	--	--
10	--	--	--	--

Table 6. For single events, we calculate that $\lambda = \frac{15}{10} = 1.5$ for columns and $\lambda = \frac{15}{5} = 3$ for rows. Values were rounded as in Table 5. We found that the probability values for 9 and 10 occurrences in columns were statistically insignificant.

Poisson Distribution of Double Events

Value of *k*	***P (X=k)* for col**	**P_k x 10 col**	***P (X=k)* for rows**	**P_k x 5 rows**
1	.2681	3	.3595	2
2	.0536	1	.1438	1
3	.0072	0	.0383	0
4	.0072	0	.0077	0
5	.0001	0	--	--
6	--	--	--	--
7	--	--	--	--
8	--	--	--	--
9	--	--	--	--
10	--	--	--	--

Table 7. For double events, we calculate that $\lambda = \frac{4}{10} = 0.4$ for columns and $\lambda = \frac{4}{5} = 0.8$ for rows. Values were rounded as in Table 5. We found that the probability values for 6 through 10 occurrences in columns and 5 for rows were statistically insignificant.

Poisson Distribution of Triple Events

Value of *k*	***P (X=k)* for col**	**P_k x 10 col**	***P (X=k)* for rows**	**P_k x 5 rows**
1	.0905	1	.1638	1
2	.0045	0	.0011	0
3	.0002	0	.0001	0
4	0	0	--	--
5	--	--	--	--
6	--	--	--	--
7	--	--	--	--
8	--	--	--	--
9	--	--	--	--
10	--	--	--	--

Table 8. For triple events, we calculate that $\lambda = \frac{1}{10} = 0.1$ for columns and $\lambda = \frac{1}{5} = 0.2$ for rows. Values were rounded as in Table 5. We found that the probability values for 5 through 10 occurrences in columns and 4 and 5 in rows were statistically insignificant.

Summary of Event Distribution

Event Type	**Avg # per column**	**Avg # per row**	**Total #**
Zero	3.33	4.50	30
Single	1.71	3.50	15
Double	0.13	0.25	4
Triple	0.10	0.20	1
TOTAL	--	--	50

Table 9. Summary of the average numbers of event densities per column and per row for the matrices N_i.

Results

First, we display the results of the methods described in (1) and (2) from the previous section:

Instrumentation for Matrices

Trial	I	II	III	IV	V
1	IV	III	I	II	VI
2	IV	V	II	III	VI
3	II	VI	IV	III	V

Table 10. A fair die was rolled five times per score to determine the respective instrumentations for rows I-V, as indicated in Table 3.

Time Intervals for Matrices

Trial	1	2	3	4	5	6	7	8	9	10
1	48	44	39	47	37	59	22	40	26	28
2	2	33	25	57	53	14	41	50	46	36
3	37	21	34	59	52	19	25	41	32	10

Table 11. For each score, 10 random integers between 1 and 60 were generated to represent the time intervals (in seconds) of each column of the matrices N_i.

Durations of Scores

Matrix	Average Col Duration	Total Duration
N_1	39 secs	6 mins 30 secs
N_2	35.7 secs	5 mins 57 secs
N_3	33 secs	5 mins 30 secs

Table 12. A comparison of average column duration and total duration of each score, assuming that at least one player never skips a column.

The resulting scores appear in Appendix A of this document.

Discussion

As we observed in the calculations for the Poisson distributions for zero, single, double, and triple events, the need to round to the nearest integer greatly impacts the resulting number of occurrences of each event across the 50 cells, particularly as a result of the smaller sample size than in Xenakis's original 196-cell matrix. As such, future implementations should strongly consider using a larger sample size, permitting availability of personnel and instrumentations.

In practice, the allowance for performers to proceed to a following column at their own volition only impacted decision making when players believed other players would not reciprocate. It is possible that this decision was also made on the basis of the ability to sound over other timbres; because of this, future studies should examine the distribution of sound frequencies and amplitudes for each timbre. While the instrumentation differed between scores, there was never an instance in which one timbre was doubled, which may also have yielded interesting results regarding frequency and amplitude of individual players' sounds.

Further distinctions should be made by examining performances in which (1) players are not able to perceive one another (i.e. individual tracks are recorded one at a time), (2) players have access to the entire score but not the sound produced by other players, (3) players have access to the sound produced by other players in real time, but not to the score, and (4) players have complete access at all points in time, as in this study.

Future studies might also consider incorporating a payoff matrix into the schematic to provide further opportunity for strategy-making on the part of the performers. Additionally, one might consider comparing the performance results of a scaled matrix (with columns scaled according to respective time intervals) to that of an unscaled matrix as presented in this study.

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Appendix A

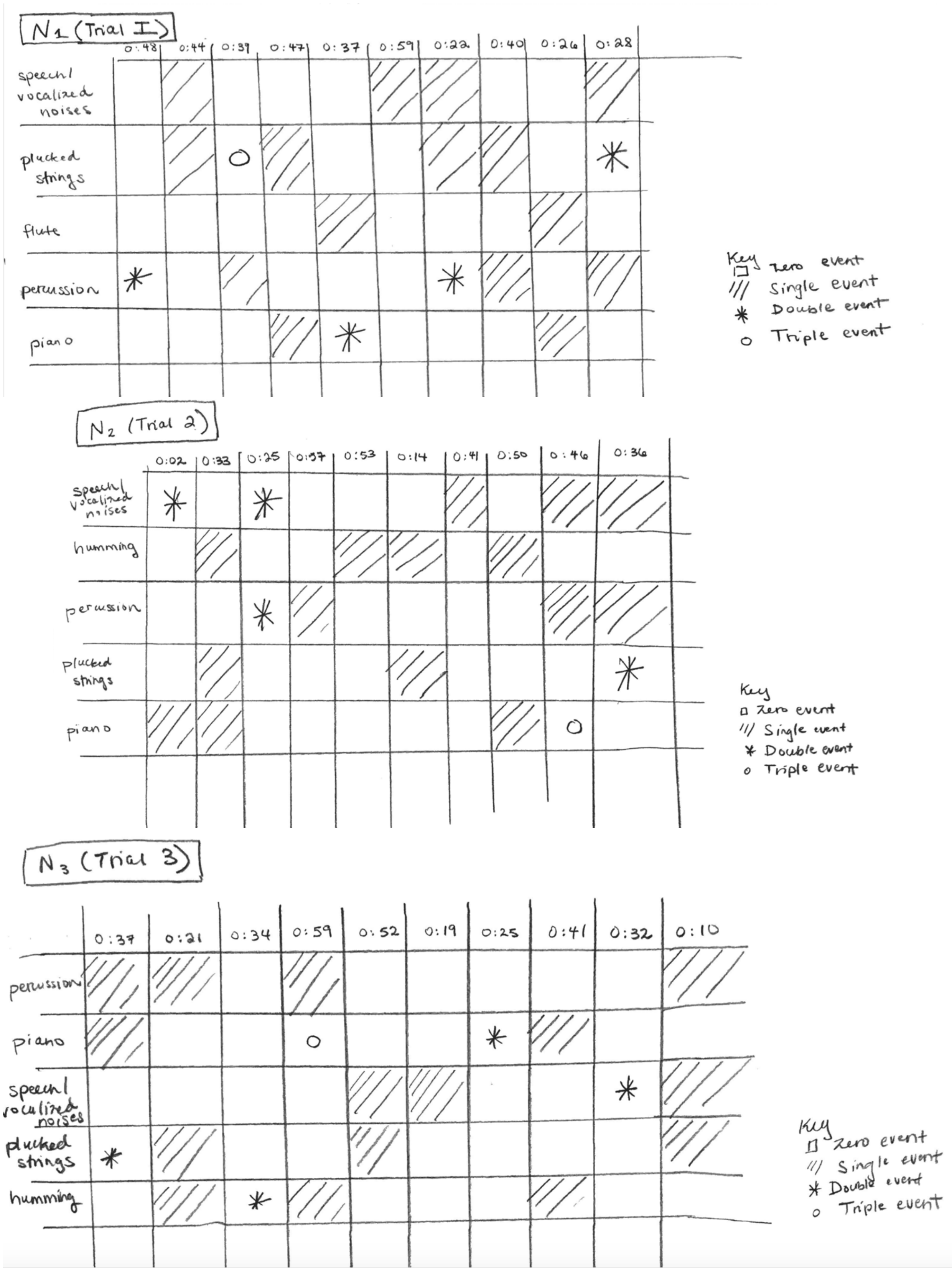

Figure A1. Respectively, the matrices N_1, N_2, and N_3 generated by the methods of this study.